Assignment - 02

Q.2 solve $y \ln y \, dx + (x - \log_e y) dy = 0$. —— (1)

Sol^n: Here $M = y \log_e y$, $N = (x - \log_e y)$

Partially diff. M w.r.t y and N w.r.t x

$\boxed{\frac{\partial M}{\partial y} = y \cdot \frac{1}{y} + \log_e y} = 1 + \log_e y$

$\cancel{\frac{\partial N}{\partial x} = \left(1 - \frac{1}{y}\right)}$ Here $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ eq^n (1) is nonexact diff. eq^n

$\boxed{\frac{\partial N}{\partial x} = 1}$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1 - 1 - \log_e y}{y \log_e y} = \frac{-\log_e y}{y \log_e y} = -\frac{1}{y},$$ it being a function of y. Hence I.F $= e^{\int -1/y \, dy} = \boxed{\frac{1}{y} = \text{I.F}}$

Now integrating factor multiplied by eq^n (1)

$(y \log_e y) \frac{1}{y} dx + (x - \log_e y) \frac{1}{y} dy = 0$ where $M' = \log y$ & $N' = \frac{x}{y} - \frac{\log_e y}{y}$

$\log_e y \, dx + \left(\frac{x}{y} - \frac{\log_e y}{y}\right) dy$ —(2) $\frac{\partial M'}{\partial y} = \frac{1}{y}$, $\frac{\partial N'}{\partial x} = \frac{1}{y}$

Hence eq^n (2) becomes exact diff eq^n; therefore sol^n is of eq^n (2) is

$$\int_{y=\text{const}} M \, dx + \int (\text{the terms of } N \text{ not containing } x) \, dy = c$$

$$x \log_e y - \int \frac{\log_e y}{y} dy = c$$

$$\boxed{x \log_e y - \frac{1}{2}(\log_e y)^2 = c}$$ ~~sol~~ Required sol^n

Q.3 solve initial value problem $\frac{dy}{dx} = \frac{2}{x} y + x$, $y(1) = 2$.

Solution The given diff eq^n can be written as

$\frac{dy}{dx} - \frac{2}{x} y = x$ it is a linear diff eq^n of the form

$\frac{dy}{dx} + Py = Q$, where $P = \frac{-2}{x}$, $Q = x$

$$\text{I.F} = e^{\int P dx} = e^{\int -\frac{2}{x} dx} = e^{-2 \log_e x}$$

$$\boxed{\text{I.F} = \frac{1}{x^2}}$$

The sol^n of above eq^n is $y \cdot \text{I.F} = \int Q \cdot (\text{I.F}) dx + C$

$$y \cdot \frac{1}{x^2} = \int x \cdot \frac{1}{x^2} dx + C \Rightarrow \frac{y}{x^2} = \int \frac{1}{x} dx + C$$

$$\frac{y}{x^2} = \log_e x + c \text{ or } \boxed{y = x^2 \log_e x + c \cdot x^2} \text{ —(1)}$$

put $x = 1$, and $y = 2$ in above eq^n (1)

$2 = \log_e(1) + c. \Rightarrow \boxed{c = 2}$

$$\boxed{y = x^2 \log_e x + 2x^2} \text{ OR } \boxed{y = x^2 (\log_e x + 2)}$$

Assignment-02

04) Solve $\frac{dy}{dx} + \frac{y}{x}\log_e y = \frac{y}{x^2}(\log_e y)^2$ —(1)

Soln: The given diff eqn is a ~~Bou~~ Bernoulli's eqn in which eqn (1) divided by $y(\log_e y)^2$ both the sides then it becomes

$$y^{-1}(\log_e y)^{-2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{\log_e y} = \frac{1}{x^2} \quad —(2)$$

put $\frac{1}{\log_e y} = z$ therefore $\frac{-1}{(\log_e y)^2}\cdot\frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$

putting all the values in eqn (2) then it becomes

$$-\frac{dz}{dx} + \frac{z}{x} = \frac{1}{x^2} \quad \text{or} \quad \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2} \quad —(3)$$

eqn (3) becomes linear eqn in the terms of z

so that $I.F = e^{\int -\frac{1}{x}dx} = e^{-\log_e x} = \boxed{\frac{1}{x} = I.F}$

and solution of eqn (3) becomes

$$z\cdot(I.F) = \int -\frac{1}{x^2}\cdot(I.F)\,dx + c$$

$$\frac{z}{x} = -\int \frac{1}{x^3}dx + c \Rightarrow \frac{z}{x} = -\frac{x^{-3+1}}{(-3+1)} + c$$

$$\frac{z}{x} = \frac{1}{2x^2} + c \quad \text{or} \quad \boxed{z = \frac{1}{2x} + cx}$$

But $z = \frac{1}{\log_e y}$

$$\boxed{\frac{1}{\log_e y} = \frac{1}{2x} + cx}$$ which is required soln.

05) Solve $x^3\frac{dy}{dx} - x^2 y = -y^4\cos x$

The above eqn can be written as

$$\frac{dy}{dx} - \frac{1}{x}y = \frac{-y^4\cos x}{x^3}$$

it is Bernoulli's eqn, Hence eqn is divided by y^4 both the sides.

$$y^{-4}\frac{dy}{dx} - \frac{1}{x}y^{-3} = -\frac{\cos x}{x^3} \quad —(2)$$

put $y^{-3} = z \Rightarrow -3y^{-4}\frac{dy}{dx} = \frac{dz}{dx}$ or $\boxed{y^{-4}\frac{dy}{dx} = \frac{-1}{3}\frac{dz}{dx}}$

putting in eqn (2)

$$\frac{-1}{3}\frac{dz}{dx} - \frac{1}{x}z = -\frac{\cos x}{x^3} \Rightarrow \frac{dz}{dx} + \frac{3z}{x} = \frac{\cos x}{x^3} \quad —(3)$$

Equation (3) becomes linear eqⁿ in the terms of Variable z. therefore I.F is

$$I.F = e^{\int \frac{3}{x} dx} = e^{3\log_e x} = x^3$$

and solution of eqⁿ is

$$z.(I.F) = \int \frac{\cos x}{x^3}.(I.F)\,dx + C$$

$$z.x^3 = \int \frac{\cos x}{x^3} \times x^3\,dx + C \Rightarrow z.x^3 = \int \cos x + C$$

OR $z.x^3 = \sin x + C$

putting $z = y^{-3}$

$$\boxed{\left(\frac{x}{y}\right)^3 = \sin x + C}$$ which is required solⁿ.

(09) find the general solⁿ $y'' - 3y' + 2y = 0$.

solⁿ: Aux. Eqⁿ of 2nd order linear eqⁿ is

$m^2 - 3m + 2 = 0 \Rightarrow m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$m = 1$ & $m = 2$ Roots of Aux eqⁿ is are Real and distinct therefore solⁿ

$\boxed{y(x) = c_1 e^{2x} + c_2 e^{x}}$, when c_1 & c_2 are constants.

(10) find the solution of initial value problems

$y'' - 2y' + 3y = 0 \quad y(0) = 1, \; y'(0) = 4$

solution:- Aux. eqⁿ of given linear eqⁿ is

$m^2 - 2m + 3 = 0$

$$m = \frac{2 \pm \sqrt{4 - 12}}{2} \Rightarrow \frac{2 \pm 2\sqrt{2}\,i}{2} = 1 \pm \sqrt{2}\,i = \alpha \pm i\beta$$

since roots of Aux eqⁿ is complex Number therefore solⁿ is $y(x) = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$, where $\alpha = 1$, $\beta = \sqrt{2}$

$$y(x) = e^{x}[c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x] \quad \text{---} (1)$$

Apply initial conditions.

$y(0) = 1 = [c_1(1) + c_2(0)] \quad \because e^0 = 1$

$\boxed{c_1 = 1}$

diff eqⁿ ① w.r.t x.

$$y'(x) = e^x\left[-\sqrt{2}\,c_1 \sin\sqrt{2}x + \sqrt{2}\,c_2 \cos\sqrt{2}x\right] + \left[c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x\right]$$

Apply 2nd I.V. conditions.

$y'(0) = 4 = \sqrt{2}\,c_2 + c_1$ ∵ But $c_1 = 1$

$$\boxed{c_2 = \frac{3}{\sqrt{2}}}$$

putting the value of c_1 & c_2 in eqⁿ ②

$$\boxed{y(x) = e^x\left[\cos\sqrt{2}x + \frac{3}{\sqrt{2}}\sin\sqrt{2}x\right]}$$

(06) Define the wronskian $W(y_1, y_2)$ of any two diff. function y_1 & y_2 defined interval $(a,b) \subseteq R$. show that $W(y_1, y_2) = 0$ if y_1 and y_2 are Linearly dependent.

Solution :- Let y_1 and y_2 are two differentiable function defined in an interval $(a,b) \subseteq R$. then wronskian of y_1 & y_2 is defined as follows:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

where y_1' and y_2' are derivative of y_1 & y_2 respectively.

Note that: The determinate of wronskian is not zero at some to then it is linearly Independent ~~other~~ otherwise it is linearly dependent.

Ex: Let $y_1(t) = 2\sin^2 t$ and $y_2(t) = 1 - \cos^2 t$

$y_1'(t) = 4\cdot\sin t\cdot\cos t$ $\quad$ $y_2'(t) = +2\sin t\cdot\cos t = \sin 2t$

$= 2\sin 2t$

$$W(y_1, y_2) = \begin{vmatrix} 2\sin^2 t & 1-\cos^2 t \\ 2\sin 2t & \sin 2t \end{vmatrix}$$

$$= \begin{vmatrix} 2\sin^2 t & \sin^2 t \\ 2\sin 2t & \sin 2t \end{vmatrix} = 0 \text{ for value of } t$$

Hence $y_1(t)$ and $y_2(t)$ are Linearly dependent.